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Corresponding to $(x_1^2 + x_2^2)^2 = (x_1^2 - x_2^2)^2 + 4x_1^2x_2^2$, we have

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 = (x_1^2 - x_2^2 + x_3^2 - x_4^2)^2 + 4(x_1^2 + x_3^2)(x_2^2 + x_4^2)$$

= the sum of three squares.

$$(x_1^2 + x_2^2 + \cdots + x_8^2)^2$$
$$= (x_1^2 - x_2^2 + - + \cdots - x_8^2)^2 + 4(x_1^2 + x_3^2 + x_5^2 + x_7^2)(x_2^2 + x_4^2 + x_6^2 + x_8^2)$$

= the sum of five squares, since by Euler's Identity the last term can be expressed as the sum of four squares.

Similarly,

$$(x_1^2 + x_2^2 + \cdots + x_{16}^2)^2$$
$$= (x_1^2 - x_2^2 + - + \cdots - x_{16}^2)^2 + 4(x_1^2 + x_3^2 + \cdots + x_{15}^2)(x_2^2 + x_4^2 + \cdots + x_{16}^2)$$

= the sum of nine squares.

The fact that the product of two numbers each of which is the sum of eight squares is itself the sum of eight squares is stated, with a reference to "Thomson, 1877," on page 115 of Carmichael's *Diophantine Analysis*, 1915.

An identity for the eight-square case is as follows:

$$\begin{aligned}&(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2) \times (p^2 + q^2 + r^2 + s^2 + t^2 + u^2 + v^2 + w^2)\\ &\quad\equiv (ap + bq + cr + ds + et + fu + gv + hw)^2 + (aq - bp + cv - du + ew + fs - gr - ht)^2\\ &\qquad + (ar - bv - cp - dt + es - fw + gq + hu)^2 + (as + bu + ct - dp - er - fq - gw + hv)^2\\ &\qquad + (at - bw - cs + dr - ep + fv - gu + hq)^2 + (au - bs + cw + dq - ev - fp + gt - hr)^2\\ &\qquad + (av + br - cq + dw + eu - ft - gp - hs)^2 + (aw + bt - cu - dv - eq + fr + gs - hp)^2.\end{aligned}$$

NOTE (written after the above was in type)—Euler's identity given in a letter to Goldbach dated May 4, 1748 (*Correspondance Mathématique et Physique* ed. by Fuss, vol. 1, 1843, p. 452) was as follows: "Si $m = aa + bb + cc + dd$ et $n = pp + qq + rr + ss$ erit $mn = A^2 + B^2 + C^2 + D^2$ existente $A = ap + bq + cr + ds$, $B = aq - bp - cs + dr$, $C = ar + bs - cp - dq$, $D = as - br + cq - dp$." This was generalized by C. F. Degen in 1822 (*Mém. Acad. Sc. St. Pétersbourg*, vol. 8 [1817–18], pp. 207–219) as follows:

$$\begin{aligned}&(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2) \times (p^2 + q^2 + r^2 + s^2 + t^2 + u^2 + v^2 + w^2)\\ &\quad\equiv (ap + bq + cr + ds + et + fu + gv + hw)^2 + (aq - bp + cs - dr + eu - ft + gw - hv)^2\\ &\qquad + (ar - bs - cp + dq \mp ev \pm fw \pm gt \mp hu)^2 + (as + br - cq - dp \pm ew \pm fv \mp gu \mp ht)^2\\ &\qquad + (at - bu \pm cv \mp dw - ep + fq \mp gr \pm hs)^2 + (au + bt \mp cw \mp dv - eq - fp \pm gs \pm hr)^2\\ &\qquad + (av - bw \mp ct \pm du \pm er \mp fs - gp + hq)^2 + (aw + bv \pm cu \pm dt \mp es \mp fr - gq - hp)^2.\end{aligned}$$

Cf. L. E. Dickson, "On quaternions and their generalization and the history of the eight square theorem," *Annals of Mathematics*, second series, volume 20, p. 164, 1919.

2840 [1920, 274]. Proposed by NORMAN ANNING, University of Michigan.

It is observed in a table of values of

$$\log_{10} (\operatorname{colog}_{10} x)$$

that second differences are zero for values of x in the neighborhood of 0.37. Prove that this must be the case. (Cf. Chappell's *Five-Figure Mathematical Tables*, Edinburgh, 1915, p. 180.)

SOLUTION BY C. C. WYLIE, University of Illinois.

Let $y = \log_{10} (\operatorname{colog}_{10} x)$ and $M = \log_{10} e$. Then

$$y = M \log \left(\log \frac{1}{x}\right) + M \log M.$$

$$\frac{dy}{dx} = -\frac{M}{x \log \frac{1}{x}}, \quad \frac{d^2y}{dx^2} = \frac{M\left(\log \frac{1}{x} - 1\right)}{\left(x \log \frac{1}{x}\right)^2} = 0,$$

if $x = 1/e = 0.368$ approx. Therefore, the second differences in a table of values of this function must be practically zero for values of x near 0.368.[1]

Also solved by H. L. Olson, H. S. Uhler, A. H. Wilson, and the Proposer.

2842 [1920, 274]. Proposed by H. S. UHLER, Yale University.

Express explicitly the following sextic in x as the product of a quadratic and a biquadratic:

$$3x^6 - 6k_1x^5 + (7k_1^2 - 9k_2^2)x^4 - 2(2k_1^3 - 4k_1k_2^2 - 3k_3^3)x^3 + [(k_1^2 - k_2^2)^2 - 9k_1k_3^3]x^2$$
$$- (k_1^2 - 2k_2^2)(k_1k_2^2 - 9k_3^3)x + (k_1^2 - 3k_2^2)(k_2^4 - 3k_1k_3^3).$$

I. Solution by W. D. Cairns, Oberlin College.

If $k_1 = k_3 = 0$, the expression can be factored thus:

$$(3x^4 + k_2^4)(x^2 - 3k_2^2).$$

If $k_1 = k_2 = 0$, it can be factored thus:

$$3x^3(x^3 + 2k_3^3) \text{ or } (3x^4 + 6k_3^3x)x^2.$$

If $k_2 = k_3 = 0$, it can be factored thus:

$$x^2(3x^2 - 3k_1x + k_1^2)(x^2 - k_1x + k_1^2).$$

These will be consistent only if one factor contains, at least, the terms $3x^4 - 3k_1x^3 + k_1^2x^2 + 6k_3^3x + k_2^4$, and the second $x^2 - k_1x + k_1^2 - 3k_2^2$. The last term of the given expression shows that the first factor must contain also the term $-3k_1k_3^3$, and multiplication of these two tentative factors shows that the term $-k_1k_2^2x$ completes the first factor. Thus we have as the two factors: $3x^4 - 3k_1x^3 + k_1^2x^2 - (k_1k_2^2 - 6k_3^3)x + (k_2^4 - 3k_1k_3^3)$ and $x^2 - k_1x + (k_1^2 - 3k_2^2)$.

II. Remarks by H. P. Manning, Providence, R. I.

The polynomial can be written $u_3k_3^3 + v_6$, where u_3 and v_6 are polynomials of degrees 3 and 6 in x, k_1 and k_2. If there are rational factors k_3^3 must go entirely with one factor, since u_3 and v_6 are not the cubes of rational expressions. The factor which does not contain k_3 will be the highest common factor of u_3 and v_6. We can also factor u_3, noting that it contains only the second power of k_2.

2845 [1920, 326]. Proposed by E. L. POST, Princeton University.

Prove that if y_x is a solution of the functional equation

$$y_x = \frac{y^2_{x+1}}{x} + y_{x+1}$$

for positive integral values of x with $y_x > 0$, then

$$\lim_{x\to\infty} y_x \log x = 1.$$

Solution by the Proposer, and Otto Dunkel, Washington University.

If x and y_x are both positive, the equation

$$y_x = \frac{y^2_{x+1}}{x} + y_{x+1} \tag{1}$$

has one positive root,

$$y_{x+1} = \frac{\sqrt{x^2 + 4xy_x} - x}{2}.$$

From (1), we clearly have then that

$$y_x > y_{x+1} > 0. \tag{2}$$

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[1] Since the second difference $y(a + 2h) - 2y(a + h) + y(a) = h^2y''(\theta)$, where $a < \theta < a + 2h$.—Editors.